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Free Field Theory as a String Theory?

Rajesh Gopakumar

Harish-Chandra Research Institute, Chhatnag Rd., Jhusi, Allahabad, India 211019

Abstract

An approach to systematically implement open-closed string duality for free large N gauge theories is summarised. We show how the relevant closed string moduli space emerges from a reorganisation of the Feynman diagrams contributing to free field correlators. We also indicate why the resulting integrand on moduli space has the right features to be that of a string theory on AdS.

1. Introduction: Two Questions

The picture of 'tHooft's double line diagrams (or open string diagrams) getting glued up into closed string worldsheets in the large N limit seems to be borne out in the few examples of the gauge-string duality that we concretely understand. Making this picture precise is essential if we are to obtain, say, a string dual to realistic gauge theories. To this end we will consider a couple of questions:

– How *exactly* does a large N field theory reorganise itself into a dual closed string theory?

– Can we *systematically* construct the closed string theory starting from the field theory?

We will address these questions in the simplest of contexts: that of *free* field theories in the large N limit. Though we will be always keeping an eye on the extensibility of our results to the case of non-zero 'tHooft coupling λ, at least in a perturbative expansion.

The general expectation from gauge-string duality is that

$$\langle \mathcal{O}_1(k_1)\ldots\mathcal{O}_n(k_n)\rangle_g = \int\limits_{\mathcal{M}_{g,n}} \langle \mathcal{V}_1(k_1,\xi_1)\ldots\mathcal{V}_n(k_n,\xi_n)\rangle_{WS}. \quad (1)$$

On the left hand side, the $\mathcal{O}_i$ are gauge invariant operators and the subscript g refers to the contribution to the correlator from Feynman diagrams of genus g. Recall that the $\frac{1}{N}$ expansion helps us isolate contributions of a given genus. On the right hand side are the corresponding vertex operators $\mathcal{V}_i$ of the dual string theory. The subscript WS refers to the averaging with respect to a worldsheet sigma model action. There is then a further integration over the moduli space $\mathcal{M}_{g,n}$ of genus g Riemann surfaces with n marked punctures (labelled by the ξ_i).

Can we somehow recast the left hand side into the form we expect on the right hand side? This is, in essence, what our pair of questions amount to. Addressing the first question, we will see that there is a simple way to organise the different Feynman diagram contributions to the free field n-point correlation function so that the nett sum can be written as an integral over the moduli space of an n-punctured Riemann surface.

Email address: gopakumr@mri.ernet.in (Rajesh Gopakumar).

For simplicity of illustration, we will consider (the genus g contribution to) correlators of free scalar composites $\mathrm{Tr}\Phi^J(k)$.

$$G_g^{\{J_i\}}(k_1, k_2, \ldots k_n) = \langle \prod_{i=1}^{n} \mathrm{Tr}\Phi^{J_i}(k_i) \rangle_g. \tag{2}$$

We will write the individual Feynman diagram contributions to this correlator in Schwinger parametrised form. By reducing the original graphs to a set of "skeleton" graphs, we will argue that the sum over the inequivalent skeleton graphs together with the integral over the Schwinger parameters gives precisely a cell decomposition of the moduli space $\mathcal{M}_{g,n} \times R_+^n$. This gives a very explicit prescription on how to reorganise field theory amplitudes into string theory amplitudes.

Moreover, this prescription also gives us a handle on our second question. We will see that the Schwinger parametrised form will enable us to write the field theory amplitude Eq.(2) as

$$G_g^{\{J_i\}}(k_1, k_2, \ldots k_n) = \int\limits_{\mathcal{M}_{g,n} \times R_+^n} [d\sigma] \rho^{(J_i)}(\sigma) e^{-\sum_{i,j=1}^{n} k_i \cdot k_j g_{ij}(\sigma)}. \tag{3}$$

The σ collectively denote the moduli of $\mathcal{M}_{g,n} \times R_+^n$. The functions $\rho^{(J_i)}(\sigma)$ and $g_{ij}(\sigma)$ can be explicitly written down. In this form the integrand on moduli space is very reminiscent of string theory. In fact, as in the expressions for flat space, the exponential factor is a universal one for all correlators (not just those of these scalars). All the dependence on the J_i are in the multiplicative prefactor $\rho^{(J_i)}(\sigma)$ which in turn is independent of the momenta, for this particular class of correlators. For more general correlators, the prefactor will contain a polynomial dependence on the momenta, again as in flat space.

Our procedure thus gives a candidate for the world sheet correlator of vertex operators of the dual string theory [1]. How can we check this hypothesis given that we don't yet know how to quantise string theory in the kind of highly curved AdS backgrounds that would presumably be dual to the free field limit? We can, as of now, perform a few modest checks. Looking at the two and three point functions shows that Eq.(2) gives the corresponding correlators in AdS space in a very natural and encouraging way [2]. One would like to make consistency checks for the four (and higher) point functions as well. One very strong check would be to verify that the integrand, in these cases, satisfies the various properties that are required of a *local* correlator of vertex operators in a two dimensional quantum field theory. In particular, such a correlator must satisfy the constraints of the *worldsheet* Operator Product Expansion (OPE). This, in turn, is manifested in the factorisability of amplitudes in spacetime. In the field theory, this property is reflected in the *spacetime* OPE. We will briefly indicate some work in progress which aims to follow this logic through, and support our identification of the integrand.

There is also a more fundamental (but less precise) reason suggesting that we take the identification, of the integrand with a worldsheet correlator, seriously. As we will see, the logic that takes us from the field theory diagrams to the stringy moduli space, in fact, implements the geometry underlying open-closed string duality. In a sense, it exhibits concretely how the double line diagrams get glued up into a closed worldsheet with the holes closing up. Therefore we expect that this procedure should also be telling us the integrand on the closed string worldsheet. One would like to believe then that we have, in all the various worldsheet correlators, all the information necessary to reconstruct the closed string theory. Future work will determine how far we can push ahead with this answer to our second question.

This being a summary we have tried not to get too much into details. Rather, we have expanded here on certain broader points. The details, together with a more complete set of references to related work and other approaches, may be found in the original papers [1][2].

2. Schwinger Parametrisation of Field Theory Amplitudes

The Schwinger parametric representation of field theory is a well studied subject. Essentially, one reexpresses the denominator of all propagators in a Feynman diagram via the identity (appropriate for Euclidean space

[1] Strictly speaking we would have to carry out the integral over the additional R_+^n moduli to obtain an integrand on $\mathcal{M}_{g,n}$. However, as we will see later, the string theory expression is also naturally extended to an integral over $\mathcal{M}_{g,n} \times R_+^n$, via a parametrisation of the n external legs of the vertex operators.

[2] Though in this case the moduli space is trivial and what we are seeing is the R_+^n factor. See previous footnote.

correlators)

$$\frac{1}{p^2} = \int_0^\infty d\tilde{\tau} \exp\{-\tilde{\tau} p^2\}. \tag{4}$$

We can apply this to the individual Feynman graphs (of genus g) that contribute to Eq.(2). We obtain

$$G_g^{\{J_i\}}(k_1, k_2, \ldots k_n) = \sum_{graphs} \int [d^d p] \int_0^\infty [d\tilde{\tau}] e^{-\tilde{P}(k,p,\tilde{\tau})}. \tag{5}$$

Here $\{p\}$ collectively denote all the independent internal momenta in the loops of the Feynman graph, and similarly, $\{\tilde{\tau}\}$ the Schwinger parameters, one for each internal edge. Since we have repeatedly used Eq.(4) in arriving at this result, it is clear that the exponent $\tilde{P}(k, p, \tilde{\tau})$ is quadratic in all the momenta (external as well as internal).

Having converted all the momentum integrals into Gaussian integrals, we can carry them out explicitly. It is a little intricate to keep track of the details of the momentum flow. But the final expressions for an arbitrary Feynman diagram can be compactly written in graph theoretic terms. For the case of scalar fields, the expressions can be looked up in field theory textbooks such as Itzykson and Zuber. The result (in d dimensions) is

$$G_g^{\{J_i\}}(k_1, k_2, \ldots k_n) = \sum_{graphs} \int_0^\infty \frac{[d\tilde{\tau}]}{\Delta(\tilde{\tau})^{\frac{d}{2}}} \exp[-P(\tilde{\tau}, k)]. \tag{6}$$

The expressions for $P(\tilde{\tau}, k)$ and $\Delta(\tilde{\tau})$ are

$$\Delta(\tilde{\tau}) = \sum_{T_1} (\prod^{l} \tilde{\tau}). \tag{7}$$

$$P(\tilde{\tau}, k) = \Delta(\tilde{\tau})^{-1} \sum_{T_2} (\prod^{l+1} \tilde{\tau})(\sum k)^2. \tag{8}$$

The sum is over various 1-trees and 2-trees obtained from the original loop diagram. A 1-tree is obtained by cutting l lines of a diagram with l loops so as to make a connected tree. While a 2-tree is obtained by cutting $l+1$ lines of the loop so as to form two disjoint trees. Eq.(7) indicates a sum over the set T_1 of all 1-trees, with the product over the l Schwinger parameters of all the cut lines. The sum over T_2 in Eq.(8) similarly indicates a sum over the set of all two trees, where the product is over the $\tilde{\tau}$'s of the $l+1$ cut lines. And $(\sum k)$ is understood to be the sum over all those external momenta k_i which flow into (either) one of the two trees. (Note that because of overall momentum conservation, it does not matter which set of external momenta one chooses.)

At this stage we do not seem to have accomplished very much of a simplification, since we are left with a large number of integrals over the Schwinger parameters. In fact, since the total number of Wick contractions that contribute to Eq.(2) is $\frac{1}{2}\sum_i J_i$, there are as many propagators and therefore Schwinger parameters. If the operators we are considering have large J_i, then the corresponding number of integrals is also large. If we are to convert this into something universal for all n-point functions, we have to look for a simplification in this representation.

There is indeed such a simplification: though the integral depends naively on a large number of Schwinger parameters, the actual non-trivial dependence is only on a certain combination of them. To see this, it is best to view the Feynman diagrams as double line diagrams or "fatgraphs". Between any two of the external vertices there can be multiple propagators which are homotopically deformable into each other (i.e. without crossing other lines). Note that viewing the Feynman diagrams in the double line representation provides an ordering of edges at each vertex and we can unambiguously speak of edges which are deformable into each other. This is one of the places where the underlying non-abelian structure plays a crucial role. We will denote by m_r, the number of such legs between a fixed pair of vertices (and fixed homotopy class) labelled by r. We can then define an "effective" Schwinger parameter τ_r for this set of edges by

$$\frac{1}{\tau_r} = \sum_{\mu_r=1}^{m_r} \frac{1}{\tilde{\tau}_{r\mu_r}}. \tag{9}$$

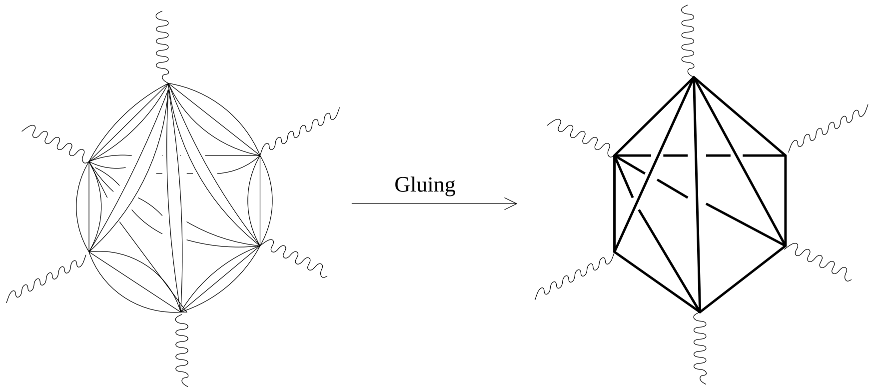

Figure 1. Fig.1: Gluing up into a skeleton graph.

The simplification is that it is this effective parameter that really enters the Schwinger parametrised expressions. It can be shown [1] that

$$P(\tilde{\tau}, k) = P_{skel}(\tau, k), \tag{10}$$

$$\Delta(\tilde{\tau}) = \frac{\prod_{r,\mu_r} \tilde{\tau}_{r\mu_r}}{\prod_r \tau_r} \Delta_{skel}(\tau). \tag{11}$$

The right hand side of these equations says that one can rewrite the original expressions, essentially, as a function of the τ_r. Moreover, the functions that appear on the RHS are defined in exactly the same graph theoretic way as in Eqs.(8)(7) except that we replace the original graph by its *skeleton graph.* The skeleton graph is obtained from the original one by gluing all the m_r homotopic edges into a single edge labelled by r. The process is illustrated for a sphere level diagram in Fig. 1. This skeleton graph is the simpler and more universal graph underlying the original Feynman graph.

We should mention that the gluing up of the Feynman diagram into a skeleton graph can be intuitively understood from a correspondence between Feynman graphs and electrical networks. (This correspondence was first pointed out in Bjorken's thesis. See [5].) The essence is that the momenta play the role of currents and the Schwinger parameters the role of resistances. As is evident from Fig. 1, the gluing up of homotopic edges and replacement by an effective parameter τ_r in Eq.(9) is nothing but parallel resistors being replaced by a single effective resistance. From the point of view of open-closed string duality, this gives us an intuitive picture of the gluing up of (some of) the holes in the original Feynman graph (or open string diagram). Note that the skeleton graph has the same genus g as the original graph.

Using Eqs. (10),(11), for a given diagram of fixed multiplicity $\{m_r\}$ and connectivity, we can rewrite the Schwinger integral over the $\tilde{\tau}$'s as one over the τ's:

$$\int_0^\infty \frac{\prod_{r,\mu_r} d\tilde{\tau}_{r\mu_r}}{\Delta(\tilde{\tau})^{\frac{d}{2}}} e^{-P(\tilde{\tau},k)} = C^{\{m_r\}} \int_0^\infty \prod_r \left(\frac{d\tau_r}{\tau_r^{(m_r-1)(\frac{d}{2}-1)}}\right) \frac{e^{-P_{skel}(\tau,k)}}{\Delta_{skel}(\tau)^{\frac{d}{2}}}. \tag{12}$$

We have also carried out the change of variables in the measure from $\tilde{\tau}$ to τ which gives rise to a purely numerical factor $C^{\{m_r\}}$. It depends only on the multiplicities and can be explicitly computed [1] but will not be important for us at the moment. Note that the details of the specific correlator, such as the J_i, are contained only in the first term (through the dependence on the m_r). P_{skel} and Δ_{skel} depend only on the topology (connectivity) of the skeleton graph and are independent of the J_i.

This universality suggests that we organise the sum over all graphs in Eq.(6) into a sum over graphs having the same underlying skeleton graph (but different multiplicities $\{m_r\}$) and then a sum over various inequivalent skeleton graphs. The first sum can be carried out explicitly, since the only dependence on m_r appears in simple prefactors. We can write the result in the schematic form (the complete answer can be found in [1])

$$G_g^{\{J_i\}}(k_1, k_2, \dots k_n) = \sum_{skel.graphs} \int_0^\infty \frac{\prod_r d\tau_r f^{\{J_i\}}(\tau)}{\Delta_{skel}(\tau)^{\frac{d}{2}}} e^{-P_{skel}(\tau,k)}. \tag{13}$$

Essentially, the $f^{\{J_i\}}(\tau)$ come from carrying out the sum over the multipicities m_r that are compatible with the same skeleton graph and the nett number of fields $\{J_i\}$. The sum in Eq.(13) is then over the various inequivalent (i.e. with inequivalent connectivity) skeleton graphs of genus g.

We have now accomplished the kind of simplification in the Schwinger representation that we were aiming for. The number of Schwinger parameters are only as many as the number of edges of the skeleton graph, a number determined only by n and g (and not the J_i), as we will soon see. By partially gluing up all the homotopic edges we have reorganised the Feynman contributions in a way which is more universal than the original diagrams. Our n-point function has been expressed as a sum of contributions from the *moduli space of skeleton graphs*. By which we mean that we are integrating over the lengths τ of the edges of the skeleton graphs as well as summing over inequivalent skeleton graphs.

3. From Skeleton Graphs to String Moduli

How can we characterise this space of skeleton graphs? Like the original graph it has n vertices and genus g. But having glued together homotopic edges, the faces of the graphs are at least triangular. In fact, for J_i greater than a minimum value (set by n), the generic face is triangular. This is because we can always add extra edges to quadrilateral or higher faces and make all faces triangular without altering the genus g. Therefore, when we have all possible wick contractions, compatible with a given genus, in a correlator such as Eq.(2), we will have all possible edges (if J_i are enough not to provide a constraint on the number of edges at a given vertex) [3] . Hence the generic skeleton graph is a triangulation of a genus g surface with n labelled vertices (of arbitrary valency). But we should remember that there will always be contributions from exceptional graphs where one or more faces are not triangular. The maximum number of Schwinger parameters τ are associated with the triangulations. The number of edges in this case, (by an application of $V - E + F = 2 - 2g$), is given by $3(n - 2 + 2g)$. As mentioned earlier, this is independent of the J_i and other details of the correlator.

This also gives the first indication of the emergence of string moduli. The number $6g - 6 + 3n$ is exactly the number of real moduli for a genus g Riemann surface with n *holes*. Separating out the the n moduli associated with the sizes of the holes gives the number of moduli of the surface with n *punctures*.

In fact, we can argue that the moduli space of skeleton graphs (of genus g with n vertices) is identical to the moduli space $\mathcal{M}_{g,n} \times R_+^n$. Consider the generic skeleton graph with triangular faces and look at its dual (in the graph theoretic sense). The dual graph has vertices associated to each face of the skeleton graph and faces associated to each original vertex. And there are dual edges, transverse to the original ones, which connect the dual vertices. From the properties of the generic skeleton graph we can conclude that its dual graph will have n (labelled) faces, $6g - 6 + 3n$ edges and cubic vertices. The trivalent vertices of the dual follow from the triangular faces of the (generic) skeleton graph. Moreover, we will associate a length $\sigma = \frac{1}{\tau} \in (0, \infty)$ ("conductance" in the electrical analogy) to each dual edge. Note that the dual graph has the same genus g as the original one (since the number of vertices and faces have simply been interchanged).

The various inequivalent triangulations are mapped to inequivalent trivalent graphs. Therefore, associated to each skeleton graph of genus g with n vertices, with its set of $\{\tau\}$'s, is a trivalent graph of genus g with n faces and a set of lengths $\{\sigma\}$ for the dual edges. As one sums over inequivalent skeleton graphs, one goes over inequivalent trivalent graphs. In fact, we can now better appreciate the role of the non-generic skeleton graphs, with four-sided (or more) faces. They map onto dual graphs with quartic (or higher) vertices. Such graphs can be thought of as arising when two (or more) cubic vertices coalesce, i.e. when the length σ of the edge joining two cubic vertices goes to zero. In fact, one can continuously move from one trivalent graph to another inequivalent one by shrinking some individual edge ("s-channel") to zero size and then expanding the resultant quartic vertex in the other direction ("t-channel"). By this process, (known to mathematicians as Whitehead collapse) one can connect the different inequivalent cubic graphs.

It is a very non-trivial mathematical theorem that the space of trivalent fatgraphs of genus g with n labelled faces and a length associated to each edge is a cell decomposition of the space $\mathcal{M}_{g,n} \times R_+^n$. In other words, as we vary over the lengths of the edges as well as over the inequivalent graphs we obtain a single cover of $\mathcal{M}_{g,n} \times R_+^n$. Each inequivalent trivalent graph fills out a top-dimensional cell in this simplicial decomposition as we vary the

[3] We have implicitly assumed that the correlators we are considering are normal ordered so that there are no self contractions. See also next footnote.

σ's. The graphs with higher point vertices live on codimension one, and higher, boundaries of these cells (when one or more $\sigma \to 0$). At these boundaries the different cells match smoothly onto each other.

This theorem is based on the work of Mumford, Strebel, Penner and others and may be found in Kontsevich [6]. For a physicist, this statement may be made plausible by recalling that in cubic open string field theory, the string diagrams are made of strips of varying length meeting at cubic vertices [7]. In fact, it was argued by Giddings, Martinec and Witten [8], and later Zwiebach [9], that these diagrams give a single cover (in our case) of the moduli space, $\widetilde{\mathcal{M}}_{g,n}$ with n *holes*. The mathematical theorem quoted above indicates that actually this factors into an R_+^n for the diameters of the holes together with the space of n punctures, $\mathcal{M}_{g,n}$.

Thus we have argued that as we vary over the moduli space of skeleton graphs, we are covering the appropriate moduli space of string worldsheets. We should remark here that the process of going to the dual graph is in a sense a reflection of open-closed string duality. In going to the dual graph we are closing off the holes of the original graph (in replacing them by dual vertices) while opening up holes/punctures at the original vertices. We thus seem to be implementing open-closed string duality at least at the level of the geometry of worldsheets [4] . In fact, the graph duality and the assignment $\sigma = \frac{1}{\tau}$ is also natural from the point of view of the UV-IR connection. The UV region of the field theory $\tau \to 0$ is mapped to $\sigma \to \infty$ which can be seen to correspond to long distance (IR) propagation of the dual closed strings. This can also be seen from the expressions for the three point correlators in Sec. 5 [2].

So we now have a way to understand how free field theory diagrams reassemble themselves into closed string worldsheets. It allows us to view the expression Eq.(13) we obtained from field theory, as an integral over the string moduli space $\mathcal{M}_{g,n} \times R_+^n$. We also note that this reorganisation of Feynman diagrams can be performed for the n-point function of arbitrary gauge-invariant operators. The Schwinger parametrisation and the gluing up into skeleton diagrams is something which can be always carried out. The general expressions will be more cumbersome (expressions for general Schwinger parametrised amplitudes are available in the literature [3]) but for specific correlators we can always work them out explicitly. We should add that this reorganisation of field theory diagrams can be done for free field theory in any number of dimensions and with arbitrary matter content (thus not necessarily supersymmetric). However, we expect that the interacting theories will probably have dual string descriptions only in $d \leq 4$.

For an interacting theory much of our argument still goes through. After all the Schwinger parametrisation of amplitudes can still be carried out in the perturbative expansion in 'tHooft coupling λ, as also the simplification into skeleton graphs. The only difference is that we have additional "internal" vertices corresponding to the interactions. It suggests the appearance of the moduli space with additional punctures corresponding to the interactions. Presumably, the additional vertex operators associated to the interactions then exponentiate (when we sum over the perturbative expansion) and modify the background. So there is promise of extending this approach to the interacting case as well.

4. The Integrand on Moduli Space

Having reorganised the diagrams into a sum over worldsheets, we can take a closer look at the integrand. Since $\sigma = \frac{1}{\tau}$ is the more natural variable to describe the cells in moduli space we can rewrite Eq.(13) (dropping the *skel* subscripts) as

$$G_g^{\{J_i\}}(k_1, k_2, \dots k_n) = \sum_{skel.graphs} \int_0^\infty \prod_r d\sigma_r \frac{\hat{f}^{\{J_i\}}(\sigma) \exp\{-\hat{P}(\sigma, k)\}}{\hat{\Delta}(\sigma)^{\frac{d}{2}}}. \tag{14}$$

where

$$\hat{\Delta}(\sigma) \equiv \sum_{T_1} (\prod \sigma) = (\prod_r \sigma_r) \Delta(\tau = 1/\sigma) \tag{15}$$

and

[4] We should point out here that though we have been considering skeleton graphs with triangular faces and their duals, we also need to consider skeleton graphs with self-contractions (which are not homotopically trivial) at vertices. Because the dual graphs of the latter also appear amongst the cells of moduli space. This suggests that the prescription of normal ordering which drops such self-contractions is perhaps not the natural one from the dual string point of view.

$$\hat{P}(\sigma,k) \equiv \frac{1}{\hat{\Delta}(\sigma)}\sum_{T_2}(\prod\sigma)(\sum k)^2 = P(\tau=1/\sigma,k) \tag{16}$$

are defined in terms of the 1-trees and 2-trees of the skeleton graph as before but the product in both these definitions is over the lines that are *not* cut. Thus we have an explicit expression, in each cell of the moduli space, of the integrand. The universal functions $\hat{\Delta}(\sigma)$ and $\hat{P}(\sigma,k)$ smoothly go from one cell to another at the common cell boundary. The multiplicative factor $\hat{f}^{\{J_i\}}(\sigma)$ which contains, as before, all the information about the specific operators is, however, more sensitive to the constraints imposed by the J_i's in the original Feynman graph.

The nett result is that we can write the field theory correlator in the form mentioned in the introduction, namely, as

$$G_g^{\{J_i\}}(k_1,k_2,\dots k_n) = \int\limits_{\mathcal{M}_{g,n}\times R_+^n}[d\sigma]\rho^{(J_i)}(\sigma)e^{-\sum_{i,j=1}^n k_i\cdot k_j g_{ij}(\sigma)}. \tag{17}$$

We can write down $\rho^{(J_i)}(\sigma)$ and $g_{ij}(\sigma)$ explicitly in each cell of the moduli space [5]. We note again the form of the integrand which is very reminiscent of string theory expressions in flat space. Given that we seem to be implementing open-closed string duality, it is very natural to take the integrand seriously as a candidate for correlators in the unknown dual string theory on AdS. We will present, in the next section, some checks in this direction.

5. *AdS* Correlators

5.1. *Three Point Functions*

For the planar three point function

$$G_{g=0}^{\{J_i\}}(k_1,k_2,k_3) = \langle \mathrm{Tr}\Phi^{J_1}(k_1)\mathrm{Tr}\Phi^{J_2}(k_2)\mathrm{Tr}\Phi^{J_3}(k_3)\rangle_{g=0} \tag{18}$$

we can carry out the procedure outlined in the preceding sections. Namely, we first glue together the multiple lines joining each pair of vertices to get a skeleton graph which is (for generic J_i) a simple triangle. For this skeleton graph we can write an integral over the three effective Schwinger parameters, as in Eq.(12). In terms of the variables $\sigma=\frac{1}{\tau}$, the final expression in the case of the three point function is

$$G_{g=0}^{\{J_i\}}(k_1,k_2,k_3) = \int_0^\infty\prod_{r=1}^3 d\sigma_r\sigma_r^{(m_r-1)(\frac{d}{2}-1)+\frac{d}{2}-2}\frac{1}{\hat{\Delta}(\sigma)^{\frac{d}{2}}}\exp\{-\hat{P}(\sigma,k)\}, \tag{19}$$

where in terms of the parameters σ_i for the three edges, we have

$$\hat{\Delta}(\sigma) = \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 \tag{20}$$

and

$$\hat{P}(\sigma,k) = \frac{1}{\hat{\Delta}(\sigma)}[\sigma_1k_1^2+\sigma_2k_2^2+\sigma_3k_3^2]. \tag{21}$$

The multiplicites m_i in Eq.(19) are determined by the J_i to be: $m_i=\frac{1}{2}\sum_{k=1}^3 J_k - J_i$. In this case, since $\mathcal{M}_{0,3}$ is trivial, the integral in Eq.(19) is only over the R_+^3 factor. We can make a change of variables on these three variables to make the connection to AdS clear.

$$\frac{1}{\rho_i} = \frac{\sigma_i}{\hat{\Delta}(\sigma)} \Rightarrow \sigma_i = \frac{\rho_1\rho_2\rho_3}{(\sum_k\rho_k)\rho_i}. \tag{22}$$

This change of variables is motivated by the star-delta transformation of electrical networks. Namely, if σ_i are the conductances of a delta or triangle network, such as the one we have, then ρ_i are the conductances of the

[5] The contributions from the non-generic graphs are finite and have support on the boundaries of the cells. This is reminiscent of similar contributions in open superstrings. We thank Ashoke Sen for remarking on this similarity.

equivalent three pronged tree or star network. In other words, the ρ_i are the variables naturally parametrising the legs of the tree one obtains when one glues up the skeleton triangle graph.

Working out the details of the jacobian for this change of variables and simplifying the integrand one finds the simple form

$$G^{\{J_i\}}_{g=0}(k_1,k_2,k_3)=\int_0^\infty\prod_{i=1}^3 d\rho_i\rho_i^{\Delta_i-\frac{d}{2}-1}\frac{1}{(\sum_k\rho_k)^{\Sigma_k\frac{\Delta_k}{2}-\frac{d}{2}}}\times e^{-[\sum_{i=1}^3\frac{k_i^2}{\rho_i}]}. \tag{23}$$

We can write this equivalently as

$$G^{\{J_i\}}_{g=0}(k_1,k_2,k_3)=\int_0^\infty\frac{dt}{t^{\frac{d}{2}+1}}\int_0^\infty\prod_{i=1}^3 d\rho_i\rho_i^{\Delta_i-\frac{d}{2}-1}t^{\frac{\Delta_i}{2}}e^{-t\rho_i}e^{-\frac{k_i^2}{\rho_i}} \tag{24}$$

by exponentiating the denominator in Eq.(23). In position space, taking into account momentum conserving delta functions, this becomes

$$G^{\{J_i\}}_{g=0}(x_1,x_2,x_3)=\int_0^\infty\frac{dt}{t^{\frac{d}{2}+1}}\int d^dz\int_0^\infty\prod_{i=1}^3 d\rho_i\rho_i^{\Delta_i-1}t^{\frac{\Delta_i}{2}}e^{-\rho_i(t+(x_i-z)^2)} \tag{25}$$

$$=\int_0^\infty\frac{dt}{t^{\frac{d}{2}+1}}\int d^dz\prod_{i=1}^3 K_{\Delta_i}(x_i,z;t), \tag{26}$$

where

$$K_\Delta(x,z;t)=\frac{t^{\frac{\Delta}{2}}}{[t+(x-z)^2]^\Delta} \tag{27}$$

is the usual position space bulk to boundary propagator for a scalar field in AdS, corresponding to an operator of dimension Δ. The only slight difference is that we have parametrised the AdS radial coordinate by $z_0^2=t$. We see from Eq.(26) that the ρ_i are indeed parameters for the external legs of the AdS tree diagram. So the R^3_+ integral in the field theory has a natural counterpart on the AdS side. We can see a similar thing for the two point function as well.

It is encouraging, in trying to answer the second of our questions, that the *integrands* in these natural parametric representations match so well. Conformal invariance fixes the overall functional dependence on the positions. But this agreement at the level of integrands, *in conjunction* with our general arguments for n-point functions, indicates that we maybe on the right track.

The fact that the scalar three point function in our procedure could be written purely in terms of supergravity bulk-to-boundary propagators is probably special to this correlation function, especially since we expect the dual theory to be highly curved. We can give a heuristic argument as to why the full string correlator might simplify in this case.

Following [4][10] the vertex operator computation, in an AdS background, for an n-point function of these scalars would take the form

$$G^{\{J_i\}}_g(x_1\ldots x_n)=\int_{\mathcal{M}_{g,n}}\langle\prod_{i=1}^n K_{\Delta_i}(x_i,X(\xi_i);t(\xi_i))\rangle_{WS}. \tag{28}$$

In other words, the vertex operators $\mathcal{V}(\xi)$ are essentially the external wave functions of the particles in AdS promoted to worldsheet operators. Thus $X(\xi),t(\xi)$ denote worldsheet fields for the AdS target space. The averaging, as the subscript indicates, is over the worldsheet action for these and other fields (including ghosts which would generally also enter into the vertex operator). Using Eq.(27) we can rewrite Eq.(28) introducing parameters for the external legs as in the case of the three point function

$$G^{\{J_i\}}_g(x_1\ldots x_n)=\int_{\mathcal{M}_{g,n}}\int_0^\infty\prod_{i=1}^n d\rho_i\rho_i^{\Delta_i-1}\langle t(\xi_i)^{\frac{\Delta_i}{2}}e^{-t(\xi_i)\rho_i-\rho_i(x_i-X(\xi_i))^2}\rangle_{WS}. \tag{29}$$

In momentum space, this reads as

$$G_g^{\{J_i\}}(k_1 \dots k_n) = \int\limits_{\mathcal{M}_{g,n}} \int\limits_0^\infty \prod_{i=1}^n d\rho_i \rho_i^{\Delta_i - \frac{d}{2} - 1} e^{-\frac{k_i^2}{\rho_i}} \langle t(\xi_i)^{\frac{\Delta_i}{2}} e^{-t(\xi_i)\rho_i} e^{ik_i \cdot X(\xi_i)} \rangle_{WS}. \tag{30}$$

In the special case of the $g = 0$ three point function, we can argue that because of the worldsheet conformal invariance the positions of the three vertex operators are irrelevant. The ghost contribution cancels out the contribution from the non-zero modes of $X(\xi), t(\xi)$, so to say. Effectively, only the zero modes contribute and so we can replace the worldsheet averaging by an integral over the zero modes of X, t. This is easy to do. The zero mode for X just gives the overall momentum conserving delta function. That for t is then identical to the expression in Eq.(24). In fact, doing the t integral goes back exactly to the expression in Eq.(23) which we had obtained from the field theory Schwinger parametrisation after an appropriate change of variables on the moduli. In other words, carrying out the worldsheet averaging in Eq.(30) (for $n = 3$) gives Eq.(23).

5.2. *Higher Point Correlators*

The form of the n-point function Eq.(30) suggests a comparison with Eq.(17) obtained from our Schwinger parameter procedure. Encouraged by the explicit example of the three point function, we could try and identify the integral over the ρ_i in Eq.(30) with the R_+^n integral in Eq.(17). In fact, the interpretation of the R_+^n factor as the diameter of the holes also suggests an identification with the external leg parameters ρ_i. If this identification is correct, we would be directly obtaining the answers for the integrand of Eq.(30) from our field theory procedure.

In any case, a strong check of our conjecture is that our procedure should give for the integrand on moduli space an expression which is consistent with all the properties of a correlator of local operators in some two dimensional quantum field theory. This is a strong constraint since we know that local operators in a field theory obey an OPE. Various miraculous channel dualities (in spacetime) of string theory follow from this OPE. But on the other hand we know that the correlators in field theory reflect these channel dualities due to the *spacetime* OPE. Since the spacetime OPE is reflected in the Schwinger parametrised representation, we would like to see it translate into a worldsheet OPE. There is some indication that this is the case because the region of Schwinger parameter space which seems to contribute to terms in the spacetime OPE also seems to the region of string moduli space (via our mapping of the two) where vertex operators come together and one expects to see a worldsheet OPE. We hope to report on this in the near future.

6. Conclusions

So, is free field theory, in general, a string theory? The universal reorganisation of Feynman diagrams certainly gives a strong indication to that effect. But we will need to study the properties of the integrand on moduli space, we have obtained, better before we can give an affirmative answer. As mentioned above, the key point is to establish a worldsheet OPE.

We would also like to be able to extract useful information from this procedure. Perhaps even reconstruct the worldsheet action. This would be particularly important if we are to extend the procedure to the perturbative expansion in the 'tHooft coupling. We would like to see the spacetime perturbation theory reassemble itself into a worldsheet perturbation expansion which has the effect of changing the background.

The fact that it is the cubic open string field theory decomposition of moduli space that appears in our procedure, is perhaps a useful hint in understanding the general gauge-string correspondence. In other examples where open-closed string duality is explicitly realised, open string field theory has often made an appearance[11,12]. Perhaps what we are seeing here is a reflection of that.

These are some of the many questions thrown up by this approach, on which future work will hopefully shed light.

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